A Relation Between Choosability and Uniquely List Colorability

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Abstract

Let G be a graph with n vertices and m edges and assume that $f : V(G) \to \mathbb{N}$ is a function with $\sum_{v \in V(G)} f(v) = m + n$. We show that, if we can assign to any vertex v of G, a list L_v of size $f(v)$, such that G has a unique vertex coloring with these lists, then G is f-choosable. This implies that, if $\sum_{v \in V(G)} f(v) > m + n$, then there is no list assignment L such that $|L_v| = f(v)$ for any $v \in V(G)$, and G is uniquely L-colorable. Finally, we prove that if G is a connected non-regular multigraph with a list assignment L of edges, such that for each edge $e = uv$, $|L_e| = max\{d(u), d(v)\}$, then G is not uniquely L-colorable and we conjecture that this result holds for any graph.

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1 Introduction and Preliminaries

The list coloring problem was introduced about 25 years ago, by Vizing [12] and independently by Erdös, Rubin and Taylor [6]. From the theoretical point of view, Vizing introduced list coloring with the intention to study total coloring, while Erdös, Rubin and Taylor took their motivation from Dinitz's Conjecture on n by n matrices. Simply, a list coloring problem has as input a graph G and at each vertex of G, a list of allowable colors. The question is "Can G be properly vertex colored in such a way that each vertex takes its color from its prescribed list?" Indeed list coloring is a general version of the usual coloring problem. The question that may be arisen is whether it is guaranteed for a graph to be list colorable, when only the size of lists is known. The field list coloring started to flourish around 1990, and has attracted an increasing attention since then. In 1992, Alon and Tarsi [2] used a polynomial associated with a graph and gave sufficient conditions for choosability of a graph in terms of the existence of certain orientations on the edges of the graph. Our main result is based on this algebraic approach invented by Alon and Tarsi [2].

The uniquely colorable graphs has been studied extensively by several authors ([1], [3], [5], [8], [10], [13]). Here we investigate uniquely list colorable graphs and their relation with choosability. Consider a graph G with a given list L. We say that G is uniquely L-colorable if G has only one proper L-coloring. In this paper, using a strong algebraic method developed by Alon and Tarsi [2], we present some relations between uniquely list colorability and choosability of a graph. More precisely, we show that for a given list L, if G is a uniquely L-colorable graph with n vertices and m edges and the sum of list sizes in L is equal to $m+n$, then G is L'-colorable for any list L' with the same list sizes as L.

The graphs considered are finite, undirected and without loops. The vertex set of a graph G is referred as $V(G)$; its edge set as $E(G)$. The number of vertices of G is called the *order* of G, and will be usually denoted by n; the number of edges of G is called the size of G and will generally be denoted by m.

For a graph G, a *list assignment* L is a function that assigns to each vertex v of G a set L_v of colors. An *L-coloring* of G is a function c that assigns a color to each vertex of G such that $c(v) \in L_v$ for all $v \in V(G)$ and $c(u) \neq c(v)$ whenever u, v are adjacent in G. If G admits an L-coloring, then G is *L-colorable.* G is said to be *uniquely L-colorable*, if there is exactly one L-coloring. Given a function $f : V(G) \rightarrow \mathbb{N}$, we say that G is *f-choosable* if

G is L-colorable for every list assignment L satisfying $|L_v| = f(v)$ for all $v \in V(G)$. The terms *edge-list assignment, L-edge-coloring, L-edge-colorable, uniquely L-edge-colorable and f-edge-choosable* are defined analogously.

The *graph polynomial* $f_G(x_1, \ldots, x_n)$ of a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is defined by

$$f_G(x_1, \ldots, x_n) = \prod\{(x_i - x_j) | i < j, v_i v_j \in E(G)\}$$

(see [2]). Let D be an orientation of G. An oriented edge (v_i, v_j) of G is said to be *decreasing* if $i > j$. The orientation D is called *even*, if it has an even number of decreasing edges, otherwise, it is called *odd*. For non-negative integers $d_1, \ldots, d_n$, let $\mathsf{DE}(\mathsf{d_1}, \ldots, \mathsf{d_n})$ and $\mathsf{DO}(\mathsf{d_1}, \ldots, \mathsf{d_n})$ denote, respectively, the sets of all even and odd orientations of G, in which the outdegree of the vertex v_i is d_i for $1 \leq i \leq n$. Alon and Tarsi [2] find an interesting description of graph colorability in terms of the graph polynomial. In [2] they prove the following lemma.

Lemma **A.** *In the above notation*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \geq 0} (|\mathsf{DE}(\mathsf{d_1}, \ldots, \mathsf{d_n})| - |\mathsf{DO}(\mathsf{d_1}, \ldots, \mathsf{d_n})|) \Pi_{i=1}^{n} {x_i}^{d_i}$$

2 An Algebraic Approach to Choosability

In this section, based on the algebraic technique developed by Alon and Tarsi in [2], a relation between choosability and uniquely list colorability is established. As a main result, we prove that if a graph G of order n and size m is uniquely L-colorable for a list assignment L such that $|L_v| \geq 1$ for all $v \in V(G)$ and $\sum_v |L_v| = m + n$, then G is f-choosable for all functions f satisfying $f(v) = |L_v|$ for all $v \in V(G)$. First, we prove the following algebraic lemma.

Lemma **1.** *Let F be a field and let $P = P(x_1, \ldots, x_n)$ be a polynomial in n variables over F such that $deg_{x_i}(P) \leq d_i$ for $1 \leq i \leq n$. Furthermore, for $1 \leq i \leq n$, let S_i be a subset of F consisting of $d_i + 1$ elements and let $a_i \in S_i$. Suppose that $P(a_1, \ldots, a_n) \neq 0$ and $P(x_1, \ldots, x_n) = 0$ for every $(x_1, x_2, \ldots, x_n) \in \Pi_{i=1}^{n} S_i \setminus \{(a_1, \ldots, a_n)\}$. Then,*

$$P(x_1, \ldots, x_n) = c \Pi_{j=1}^{n} \Pi_{s \in S_j \setminus \{a_j\}} (x_j - s)$$

for some constant $c \in F$ with $c \neq 0$.

Proof. Let $l = \Pi_{j=1}^{n} \Pi_{s \in S_j \setminus \{a_j\}} (a_j - s)$ and define the following polynomial,

$$Q(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) - l^{-1} P(a_1, \ldots, a_n) \Pi_{j=1}^{n} \Pi_{s \in S_j \setminus \{a_j\}} (x_j - s).$$

Clearly $deg_{x_i}(Q) \leq d_i$, for any i, $1 \leq i \leq n$ and for each $(r_1, \ldots, r_n) \in \Pi_{i=1}^{n} S_i$, $Q(x_1, \ldots, x_n) = 0$. Now the result follows from Lemma 2.1 of [2]. $\Box$

We now use Lemma 1 to obtain a relation between choosability and uniquely list colorability.

THEOREM **1.** *Let G be a graph on a set $V = \{v_1, \ldots, v_n\}$ of $n \geq 1$ vertices. For $1 \leq i \leq n$, let L_{v_i} be a list of $d_i + 1$ colors where $d_i \geq 0$ is a given integer. Suppose that G is uniquely L-colorable and $d_1 + \cdots + d_n = m$ where m is the size of G. Then the following statements hold: (i) $|\mathsf{DE}(\mathsf{d_1}, \mathsf{d_2}, \ldots, \mathsf{d_n})| \neq |\mathsf{DO}(\mathsf{d_1}, \ldots, \mathsf{d_n})|$, and (ii) G is f-choosable provided that $f(v_i) = d_i + 1$ for $1 \leq i \leq n$.*

Proof. That (i) implies (ii) follows from a result of Alon and Tarsi in [2]. Hence, it is sufficient to prove (i). Let

$$K = |\mathsf{DE}(\mathsf{d_1}, \ldots, \mathsf{d_n})| - |\mathsf{DO}(\mathsf{d_1}, \ldots, \mathsf{d_n})|.$$

To show that $K \neq 0$, we use the idea of Alon and Tarsi [2]. Let

$$f_G(x_1, \ldots, x_n) = \Pi\{(x_i - x_j) | i < j, v_i v_j \in E(G)\}$$

be the graph polynomial of G. The polynomial f_G is homogeneous and every monomial of f_G has degree m. For $1 \leq i \leq n$, let $S_i = L_{v_i}$. Furthermore, let $S = \Pi_{i=1}^{n} S_i$. Since G is uniquely L-colorable, there is exactly one n-tuple $\underline{a} = (a_1, \ldots, a_n) \in S$ such that

1. $f_G(a_1, \ldots, a_n) \neq 0$ and $f_G(x_1, \ldots, x_n) = 0$ for every $(x_1, \ldots, x_n) \in S \setminus \{\underline{a}\}$.

   Now we use the same argument as in the proof of Theorem 2.1 by Alon and Tarsi in [2]. This argument implies that there is a polynomial $\bar{f}_G(x_1, \ldots, x_n)$ satisfying the following conditions.

2. $\bar{f}_G(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in S$.

3. $deg_{x_i}(\bar{f}_G) \leq d_i$, for $1 \leq i \leq n$.

4. The coefficient of $\Pi_{i=1}^{n} x_i^{d_i}$ in $\bar{f}_G$ is equal to its coefficient in f_G.

By Lemma A and 4, it then follows that the coefficient of $\Pi_{i=1}^n x_i^{d_i}$ in $\bar{f}_G$ is equal to K. Combining 1, 2, 3, and Lemma 1, we then conclude that $K \neq 0$. This completes the proof. $\square$

The above theorem has an immediate corollary.

COROLLARY **1.** *Let G be a graph of order n and size m. If $f : V(G) \to I\!N$ is a function with $\sum_{v \in V(G)} f(v) > m+n$, then there is no list assignment L such that $|L_v| = f(v)$ for any $v \in V(G)$, and G is uniquely L-colorable.*

Proof. Suppose, by contradiction, that there is such a list assignment L. Assume that the color of vertex $v \in V(G)$ in the unique coloring is c_v. We can omit a number of colors of some lists in L, such that the new list of vertex $v \in V(G)$ $(1 \leq i \leq n)$, say L'_v, contains the color c_v, and further $\sum_{v \in V(G)} |L'_v| = m + n$. Clearly G is uniquely L'-colorable. Thus, if we define $f' : V(G) \to I\!N, f'(v) = |L'_v|$, by Theorem 1, G is f'-choosable. Since $\sum_{v \in V(G)} f(v) > m+n$, we can find a vertex $v \in V(G)$ and a color $a \in L_v \backslash L'_v$. Since G is f'-choosable, if we change the color c_v in L'_v to color a, then we obtain a proper coloring of vertices of G in which the color of each vertex is from its list, and the color of vertex v is not c_v. This contradicts the uniqueness of the coloring c. $\square$

The following theorem shows that the above result is tight.

THEOREM **2.** *For any $t \in I\!N$, there exists a graph G of order n and size m and a function $f : V(G) \to I\!N$, with the following properties: (i) For any vertex v, $f(v) \geq t$, (ii) $\sum_{v \in V(G)} f(v) = m + n$, (iii) There exists a list assignment L with $|L_v| = f(v)$ for all $v \in V(G)$ such that G is uniquely L-colorable, (iv) G is f-choosable.*

Proof. For the proof, consider the complete graph K_{2t-1}, with the vertex set $\{u_1, \ldots, u_t, v_1, \ldots, v_{t-1}\}$. For each $i, j, 1 \leq i \leq t, 1 \leq j \leq t-1$, assign to u_i a list $L_{u_i} = \{1, \ldots, t\}$ and to v_j a list $L_{v_j} = \{1, \ldots, t+j\}$. By adding $t-1$ independent new vertices $\{w_1, \ldots, w_{t-1}\}$ to the complete graph K_{2t-1} and joining the vertex w_i $(1 \leq i \leq t-1)$ to all vertices $\{v_1, \ldots, v_{t-1}\}$ and $\{u_{i+1}, \ldots, u_t\}$, we get a graph G of order $3t-2$ and size $\binom{2t-1}{2} + (t-1)^2 + 1 + 2 + \cdots + (t-1)$. For each $i, 1 \leq i \leq t$, we put $L_{w_i} = \{t+1, \ldots, 2t-1\} \cup \{i\}$. We claim that G satisfies the desired properties of the theorem. First, we show the uniqueness of the coloring. Since all colors $\{1, \ldots, t\}$ appear in the vertices $\{u_1, \ldots, u_t\}$ and v_1 is adjacent to these vertices, v_1

can only be colored by $t+1$. Furthermore, since v_2 is adjacent to the vertices $\{u_1, \ldots, u_t, v_1\}$, its color should be $t+2$. Similarly the color of any vertex v_i should be $t+i$, for any i, $1 \leq i \leq t-1$. On the other hand since w_i is adjacent to all vertices $\{v_1, \ldots, v_{t-1}\}$ and $L_{w_i} = \{t+1, \ldots, 2t-1\} \cup \{i\}$, thus its color should be i. Also w_1 is adjacent to the vertices $\{u_2, u_3, \ldots, u_t\}$ and one of the vertices $\{u_1, \ldots, u_t\}$ should be colored by 1, therefore, the color of u_1 is 1. Analogously for any $j, 1 \leq j \leq t$, the vertex u_j can only be colored by j and, therefore, G has a unique vertex coloring with the given lists. It is easy to check that $\sum_{v \in V(G)} f(v) = |V(G)| + |E(G)|$. Now by Theorem 1, G is f-choosable. $\square$

It is known that, for $r = 2^{k-2}$, the complete bipartite graph $K_{r,r}$ is k-choosable [6]. This shows that, for any positive numbers $c, \epsilon < 1$, there exists a graph G of order n and size m, and a function $f : V(G) \rightarrow \mathbb{N}$, such that $\sum_{v \in V(G)} f(v) < \frac{cm}{n^\epsilon} + n$, and G is f-choosable.

3 The Non-Uniqueness of Edge Colorings

In this section, we study list edge-colorings of graphs. First, we need some further notation. The *line graph* of an arbitrary multigraph G, written $L(G)$, is the simple graph whose vertices are the edges of G, and two vertices are joined by an edge if and only if the corresponding edges of G share an endpoint. Clearly, for an edge-list assignment L of G, the graph G is (uniquely) L-edge-colorable if and only if $L(G)$ is (uniquely) L-colorable. First, we apply our results from Section 2 to prove the following theorem.

Theorem **3.** *Let G be a connected non-regular multigraph and let L be an edge list assignment of G. If for each edge $e = uv$, $|L_e| = max\{d(u), d(v)\}$, then G is not uniquely L-edge-colorable.*

Proof. Consider the line graph of G, $L(G)$. If $e = uv$ is an edge of G, then the degree of the corresponding vertex in the line graph of G is equal to $d(u) + d(v) - 2$. By assumption, we have $|L_e| = max\{d(u), d(v)\} \geq \frac{d(e)}{2} + 1$, for each $e \in E(G)$. This implies that

$$\sum_{e \in E(G)} |L_e| \geq |E(L(G))| + |V(L(G))|.$$

Since G is a connected non-regular graph, there are two adjacent vertices say, u and v, such that $d(u) \neq d(v)$. If $e = uv$, $|L_e| > \frac{d(e)}{2} + 1$. Thus,

$$\sum_{e \in E(G)} |L_e| > |E(L(G))| + |V(L(G))|.$$

Therefore, using Corollary 1, $L(G)$ is not uniquely L-colorable. Hence G is not uniquely L-edge-colorable. $\square$

There is an extensive theory for edge choosability of graphs which largely lies outside the scope of this paper, for instance see [4]. In [4] it is shown that if every edge $e = uw$ of a bipartite multigraph G is assigned a list of at least $max\{d(u), d(w)\}$ colors, then G can be edge-colored with each edge receiving a color from its list. We note that for any graph G of Class 2 ($\chi'(G) = \Delta + 1$) the above claim does not hold. Furthermore, we show that this fact is not true for non-bipartite graphs of Class 1 ($\chi'(G) = \Delta$). Figure 1 depicts a non-bipartite graph with list assignment L, that satisfies the above hypothesis but is not L-edge-colorable. The list of a thick edge e is $L_e = \{1, 2, 3, 4\}$ and the list of a thin edge e is $L_e = \{1, 2, 3\}$. Then it is easy to check that the graph is not L-edge-colorable.

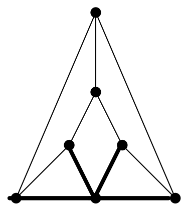

Figure 1: An example of a graph in Class $1(\chi'(G) = \Delta)$ that is not L-colorable.

Let G be a bipartite multigraph with bipartition U, W. Let $c : E(G) \rightarrow \mathbb{N}$ be a proper edge coloring of G. Here we use some definitions from [4]. We say that an edge e *sees* an edge e', if e and e' are incident with a vertex v and $c(e) > c(e')$ if $v \in U$ or $c(e) < c(e')$ if $v \in W$. If $S, T \subseteq E(G)$, we say that S *sees* T if every edge in $S \backslash T$ sees at least one edge in T. If e sees e', then we write $e \lhd e'$. In addition, we denote by $m_c(e)$ the number of edges that e sees according to the coloring c. The following lemma has been proved in [4].

LEMMA **B.** *If G is a bipartite graph and $S \subseteq E(G)$, then there is a matching $M \subseteq S$ such that S sees M.*

According to Theorem 3 of [4] we find the following result.

THEOREM **4.** *Let G be a Δ-regular bipartite multigraph ($\Delta \geq 2$), with bipartition U, W and let L be an edge-list assignment of G. If $|L_e| = \Delta$ for each edge $e \in E(G)$, then G has at least two distinct L-edge-colorings.*

Proof. Consider an arbitrary proper edge coloring c of G with Δ colors. Since the degree of any vertex of G is Δ, $m_c(e) = \Delta - 1 < \Delta = |L_e|$. First, we show that if G is an arbitrary bipartite multigraph with bipartition U, W such that $m_c(e) < |L_e|$ for each edge $e \in E(G)$, then G has an L-edge-coloring. We prove this by induction on $|E(G)|$. For any color i, $i \in \cup_{e \in E(G)} L_e$, let $S_i = \{e \in E(G), i \in L_e\}$. By Lemma B, S_i sees a matching M_i inside S_i. Now for each $e \in M_i$, we assign color i to e and for any $e \in S_i$, we remove i from L_e and reduce $|L_e|$ by one. It is not hard to see that the hypothesis ($m_c(e) < |L_e|$) in the graph $G \backslash M_i$ still remains true. The result follows by induction. Thus there is an L-edge-coloring of G, say c_L.

If we start with $S_j (j \in \cup_{e \in E(G)} L_e, j \neq i)$ instead of S_i and if M_j is the matching related to S_j, i.e. $S_j \lhd M_j$ and $M_i \cap M_j \neq \varnothing$, then we obtain two distinct L-edge-colorings. The reason is that if $e \in M_i \cap M_j$, then $c_L(e) = i$ in the first L-edge-coloring whereas $c_L(e) = j$ in the second L-edge-coloring. Thus one may assume that for any $i, j \in \cup_{e \in E(G)} L_e$, $i \neq j$, $M_i \cap M_j = \varnothing$. Now let $e = uw$ be an edge for which $c(e) = 1$. Assume that $c_L(e) = r$. We claim that for any $t \in L_e (t \neq r)$, there exists an edge e' so that e and e' are incident with a vertex $w \in W$ and $c_L(e') = t$. To see this, let M_t be the matching that S_t sees when we start with color t. We note that $e \in S_t \backslash M_t$ and by definition, e sees an edge in M_t, say, e'. Since $c(e) = 1$, we conclude that e and e' are adjacent in a vertex $w \in W$, and the claim holds. Since for any edge $a \in E(G)$ we have a proper edge coloring say c' such that $c'(a) = 1$ we conclude that, if $L_e = \{a_1, a_2, \ldots, a_\Delta\}$ then for any edge a which is incident to w we have $L_a = \{a_1, a_2, \ldots, a_\Delta\}$. By a similar argument if we interchange the colors 1 and Δ in c, we conclude that for any edge a incident to e, $L_a = \{a_1, a_2, \ldots, a_\Delta\}$. It follows that in the connected component of G containing e the lists of all edges are the same. Now by interchanging two colors of the set $\{a_1, a_2, \ldots, a_\Delta\}$ in this component, we obtain two distinct L-edge-colorings. $\square$

The following conjecture, known as *list coloring conjecture*, was formulated independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [9], [11]).

CONJECTURE **A.** *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

Galvin [7] introduced a remarkable and interesting technique and proved that Conjecture A holds for any bipartite multigraphs. The following conjecture and Theorem 3 show that the list coloring conjecture may be restated in a stronger form if G is a graph of Class 1. Also it may be considered as a complement of Theorem 3.

Conjecture. Let G be a Δ-regular graph ($\Delta \geq 2$) and let L be an edge-list assignment of G such that $|L_e| = \Delta$ for all $e \in E(G)$. Then G is not uniquely L-edge-colorable.

This is notable that by Theorem 1 the above conjecture is true for any graph of Class 2, since every graph G of Class 2 is not Δ-choosable. If this conjecture is true, using Theorem 3 we can restate Conjecture A as follows: if G is a graph of Class 1 with maximum degree Δ, then, for each edge-list assignment L satisfying $|L_e| = \Delta$ for all $e \in E(G)$, there are at least two L-edge-colorings.

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